

# COMPACT CONVEX SETS IN 2-DIMENSIONAL ASYMMETRIC NORMED LATTICES

N. JONARD-PÉREZ AND E.A. SÁNCHEZ-PÉREZ

**ABSTRACT.** In this note, we study the geometric structure of compact convex sets in 2-dimensional asymmetric normed lattices. We prove that every  $q$ -compact convex set is strongly  $q$ -compact and we give a complete geometric description of the compact convex set with non empty interior in  $(\mathbb{R}^2, q)$ , where  $q$  is an asymmetric lattice norm.

## 1. INTRODUCTION

A direct consequence of the Heine-Borel Theorem is that convex compact sets of the one dimensional Hausdorff linear spaces are the closed bounded intervals. However, compact sets of the one-dimensional, non-Hausdorff, real asymmetric normed linear space  $(\mathbb{R}, |\cdot|_a)$  have a completely different structure. The asymmetric norm  $|\cdot|_a$  is defined by

$$|r|_a := \max\{r, 0\}, \quad r \in \mathbb{R}.$$

This asymmetric norm induces a topology on  $\mathbb{R}$  in which the basic open sets are  $B_\varepsilon(r) = \{y \in \mathbb{R} : y - r < \varepsilon\}$ . It can be easily seen that in this space each convex compact set can be written either as an interval  $(a, b]$  or as an interval  $[a, b]$ , for  $-\infty \leq a \leq b < \infty$  (see Corollary 4.3 in [7]).

In the symmetric case, it is well-known that any two  $n$ -dimensional compact convex bodies are homeomorphic. Furthermore, any homeomorphism between two compact convex bodies  $A$  and  $B$  of the same dimension maps the interior of  $A$  onto the interior of  $B$  and the boundary of  $A$  onto the boundary of  $B$ .

On the other hand, it is well known that every closed convex subset  $K$  of a finite dimensional Banach space is homeomorphic to either  $[0, 1]^n \times$

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$[0, \infty)$  or  $[0, 1]^n \times \mathbb{R}^m$  for some  $n, m \in \mathbb{N}$  (see [3]). In particular, if  $K$  is unbounded and has no lines, then it is homeomorphic to  $[0, 1]^n \times [0, \infty)$ . In the special case of  $\mathbb{R}^2$  we have the following theorem (see [3, Chapter III, §6]):

**Theorem 1.** *Let  $K_1$  and  $K_2$  be two closed convex subsets of  $\mathbb{R}^2$ . Additionally, suppose that both sets have non-empty interior and none of them contains a line. If  $K_1$  and  $K_2$  are unbounded, then there exists an homeomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  sending  $K_1$  onto  $K_2$ , the interior of  $K_1$  onto the interior of  $K_2$ , and the boundary of  $K_1$  onto the boundary of  $K_2$ .*

Although some effort has been made to find an asymmetric version of the previous results (see [2, 7, 12]), at the moment there is no explicit description in the mathematical literature of the (compact) convex sets of an asymmetric normed space. The aim of this work is to provide such a description

After Corollary 11 in [12], we know that only asymmetric normed spaces not satisfying any separation axiom stronger than  $T_0$  provides a scenario that is different from the classical one for finite dimensional normed spaces. This result establishes that a finite dimensional asymmetric normed space is normable if it is  $T_1$ , and so in this case the results on representation of compact convex sets that hold are exactly the same that work in the normed case.

As in [7], we restrict our attention to the case of asymmetric norms defined using lattice norms in  $\mathbb{R}^2$ , when the canonical order is considered. We will give a geometric description of the compact convex sets in a two dimensional (real) asymmetric normed lattice. More precisely, we will show that it is possible to obtain a concrete description of all the convex bodies (compact convex sets with a non-empty interior) in the asymmetric space  $(\mathbb{R}^2, q)$ , where  $q$  is an asymmetric lattice norm.

Finally, we will use the geometric structure of compact convex sets to prove that every compact convex set in  $(\mathbb{R}^2, q)$  is strongly  $q$ -compact (see definition below).

## 2. PRELIMINARIES

The reader can find more information about asymmetric normed spaces in [1, 2, 4, 5, 6, 9, 10, 11, 12]). In this section we will only recall the basic notions about these spaces.

Consider a real linear space  $X$  and let  $\mathbb{R}^+$  be the set of non-negative real numbers. An *asymmetric norm*  $q$  on  $X$  is a function  $q : X \rightarrow \mathbb{R}^+$  such that

- (1)  $q(ax) = aq(x)$  for every  $x \in X$  and  $a \in \mathbb{R}^+$ ,
- (2)  $q(x + y) \leq q(x) + q(y)$ ,  $x, y \in X$ , and
- (3) for every  $x \in X$ , if  $q(x) = q(-x) = 0$ , then  $x = 0$ .

The pair  $(X, q)$  is called an *asymmetric normed linear space*. The asymmetric norm defines a non-symmetric topology on  $X$  that is given by the open balls  $B_\epsilon(x) := \{y \in X : q(y - x) < \epsilon\}$ ; this topology is in fact the one given by the quasi-metric  $d_q(x, y) := q(y - x)$ ,  $x, y \in X$ .

For every asymmetric normed space  $(X, q)$  the map  $q^s : X \rightarrow \mathbb{R}^+$  defined by the rule

$$q^s(x) := \max\{q(x), q(-x)\}, \quad x \in X$$

is a norm that generates a topology stronger than the one generated by  $q$ . We will use the symbols  $B_\epsilon^q$  and  $B_\epsilon^{q^s}$  to distinguish the open unit balls of  $(X, q)$  and  $(X, q^s)$ , respectively. In order to avoid confusion, we will say that a set is *q-compact* (*q<sup>s</sup>-compact*), if it is compact in the topology generated by  $q$  ( $q^s$ ). We define the *q-open* and *q-closed* (*q<sup>s</sup>-open* and *q<sup>s</sup>-closed*) sets in the same way.

Since the topology is linear, the set  $\theta_0 := \{x \in X : q(x) = 0\}$  can be used to determine when the space is not Hausdorff. This set is also relevant for characterizing compactness, and has been systematically used in [2, 12].

From an abstract point of view, the knowledge about the compact subsets of asymmetric normed linear spaces  $(X, q)$  is satisfactory in respect to their general structure ([2, 7, 12]). The finite dimensional case is in fact better known (see [7, 12]). In the case that the topology of  $(X, q)$  is  $T_1$ , then it is automatically Hausdorff –and so normed– and its properties are thus perfectly known. A more interesting case results when the topology is as weak as possible, i.e.  $T_0$ .

The canonical example of a non-Hausdorff asymmetric normed space is when the asymmetric norm is given by a lattice norm (see the example that follows Corollary 22 in [2]). Recall that a *lattice norm* in the finite dimensional space  $\mathbb{R}^n$  with the coordinatewise order is a norm  $\|\cdot\|$  that satisfies that  $\|x\| \leq \|y\|$  whenever  $|x| \leq |y|$ ,  $x, y \in \mathbb{R}^n$ . Such a norm defines an asymmetric norm  $q$  by the formula

$$q(x) := \|x \vee 0\|, \quad x \in \mathbb{R}^n,$$

where  $x \vee 0$  denotes the coordinatewise maximum of  $x$  and 0. The norm  $q$  previously defined is called an *asymmetric lattice norm*, and  $(\mathbb{R}^n, q)$  is an *asymmetric normed lattice*. A systematic study of asymmetric norms defined by means of a Banach lattice norm has been done in [7] and some of the results found there are relevant for our paper. It is well known that all norms in a finite dimensional space are equivalent,

and so all the asymmetric norms defined by means of this procedure for different lattice norms are equivalent too.

These asymmetric norms satisfy some stronger properties that are in general not true for other asymmetric norms in finite dimensional spaces. For instance, the asymmetric lattice norms are always *right bounded* with constant  $r = 1$ ; that is, for every  $\varepsilon > 0$  and  $x \in \mathbb{R}^n$ ,

$$B_\varepsilon^q(x) = B_\varepsilon^{q^s}(x) + \theta_0$$

(see [12, Definition 16] and [2, Lemma 1]).

A property that has been systematically explored in [2] for a  $q$ -compact set  $K$  is the existence of a  $q^s$ -compact *center* for  $K$ , i.e., there is a  $q^s$ -compact set  $K_0$  such that  $K_0 \subseteq K \subseteq K_0 + \theta_0$ . Sets  $K$  satisfying this property are called *strongly  $q$ -compact* ([7]).

It is not difficult to see that a set  $K$  is  $q$ -compact if and only if  $K + \theta_0$  is  $q$ -compact. Since the topology generated by  $q^s$  is finer than the one generated by  $q$ , it follows that every  $q^s$ -compact set is  $q$ -compact. This fact implies that every strongly  $q$ -compact set is  $q$ -compact. However, the converse implication is not always true. In fact, in [7, Example 4.6] we can find an example of a lattice norm  $q$  in  $\mathbb{R}^2$  and a  $q$ -compact set  $A \subset \mathbb{R}^2$  that is not strongly  $q$ -compact. In this paper, we will show that if we restrict our attention to the convex sets, it results that every  $q$ -compact convex set is strongly  $q$ -compact.

We will use standard notation. In the rest of this paper the letter  $q$  will be used to denote an asymmetric lattice norm in  $\mathbb{R}^2$ . Observe that in this case, the set  $\theta_0$  coincides with the cone  $(-\infty, 0] \times (-\infty, 0]$ . As usual, if  $A$  is a set of  $(\mathbb{R}^2, q)$ , we write  $A^\circ$  for the interior with respect to the asymmetric topology defined by  $q$  and  $\overline{A}$  for its closure. In case we want to consider these definitions for the topology defined by  $q^s$ , we will write explicitly  $A^{\circ^{q^s}}$  and  $\overline{A}^{q^s}$ . Finally, we will say that a set  $A \subset \mathbb{R}^2$  is a  *$q$ -convex body*, if  $A$  is  $q$ -compact, convex and its interior with respect to the asymmetric topology is non empty. Note that  $q$ -convex bodies are not  $q$ -closed and that they may not be  $q^s$ -closed either.

### 3. $q$ -COMPACT CONVEX SETS IN $(\mathbb{R}^2, q)$

The main purpose of this section consists on proving that every  $q$ -compact convex set in  $(\mathbb{R}^2, q)$  is strongly  $q$ -compact. We will also build some notation and preparatory results that will be used later to describe the geometric structure of  $q$ -compact convex bodies.

For the case  $n = 1$ , let us consider the topology in  $\mathbb{R}$  given by the asymmetric norm  $|\cdot|_a := \max\{\cdot, 0\}$ . We start with the following easy lemma.

**Lemma 2.** (i) *The function  $q : (\mathbb{R}^n, q) \rightarrow (\mathbb{R}, |\cdot|_a)$  is continuous.*  
(ii) *The canonical projections  $P_i((x_1, \dots, x_n)) := (0, \dots, x_i, \dots, 0)$  on  $\mathbb{R}^n$ ,  $i = 1, \dots, n$ , are continuous when considered as maps  $P_i : (\mathbb{R}^n, q) \rightarrow (\mathbb{R}^n, q)$ .*

*Proof.* For (i), use the inequality  $q(x) - q(y) \leq q(x - y)$ ,  $x, y \in \mathbb{R}^n$ . For (ii), recall that a linear map  $T : (E, p_1) \rightarrow (F, p_2)$  between asymmetric normed spaces is continuous if and only if there is a constant  $Q > 0$  such that  $p_2(T(e)) \leq Q \cdot p_1(e)$ ,  $e \in E$  (see for example [6, Sec.2.1.3]). We can assume that  $i = 1$ . Then for  $w = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$\begin{aligned} q(P_1(w)) &= \|(\max\{x_1, 0\}, 0, 0, \dots)\| \\ &\leq \|(\max\{x_1, 0\}, \max\{x_2, 0\}, \max\{x_3, 0\}, \dots)\| = q(w). \end{aligned}$$

□

Proposition 4.2 in [7] states that order bounded sets that contain their suprema are  $q$ -compact. In a sense, this is the starting point of our analysis: we use the lattice structure of the space  $(\mathbb{R}^2, q)$  to find an upper bound for its asymmetric unit ball.

Let  $K$  be a compact convex set in  $(\mathbb{R}^2, q)$ . Then the continuity of the projections together with Lemma 2 allow to define the real numbers

$$u := \sup\{P_1((x, y)) : (x, y) \in K\} \text{ and } v := \sup\{P_2((x, y)) : (x, y) \in K\}.$$

Observe that due to the compactness of  $K$ , these constants are finite and there are elements such as  $(u, y)$  and  $(x, v)$  that belong to  $K$ . This means that the sets appearing below are not empty and so there exist  $(u, \beta)$  and  $(\alpha, v)$  such that

$$\alpha := \sup\{P_1((x, v)) : (x, v) \in K\} \text{ and } \beta := \sup\{P_2((u, y)) : (u, y) \in K\}.$$

Again the continuity of the projections and the compactness of  $K$  assure that  $(u, \beta)$  and  $(\alpha, v)$  belong to  $K$ .

**Lemma 3.** *Let  $K \subseteq \mathbb{R}^2$  be a  $q$ -compact convex set such that  $(\alpha, v) \neq (u, \beta)$ . Let  $L_1 := \{(\alpha, y) : y \leq v\}$  and  $L_2 := \{(x, \beta) : x \leq u\}$ . If  $K$  has non empty interior, then the convex hull  $\text{co}(L_1 \cup L_2)$  is contained in  $K$ .*

*Proof.* As the  $q$ -interior of  $K$  is not empty, we can find an element  $(x_1, y_1)$  and an  $\varepsilon > 0$  such that  $B_\varepsilon^q((x_1, y_1)) \subseteq K$ . Since  $q$  is a lattice asymmetric norm, we find that there is a set —unbounded with respect to the norm  $q^s$ — that is included in  $B_\varepsilon((x_1, y_1))$ ; in fact, the set  $(x_1, y_1) + \theta_0$  is contained in this ball and so, also in  $K$ .

Consider the interval  $[(\alpha, v), (u, \beta)]$ . Since  $K$  is convex, we have that this interval is included in  $K$ . Taking into account that it cannot be

that  $(\alpha, v) = (u, \beta)$ , we have that  $\alpha < u$  and  $\beta < v$ . The structure of the set  $(x_1, y_1) + \theta_0$  makes it clear that there is an element  $(x_0, y_0)$  in it that satisfies  $(x_0, y_0) \leq (s, t)$  for all  $(s, t) \in [(\alpha, v), (u, \beta)]$ . Take now an element  $(\alpha, t) \in L_1$ ,  $t < v$ . By the election of  $(x_0, y_0)$  we have that there is an element  $(x_2, y_2) \in (x_0, y_0) + \theta_0 \subseteq (x_1, y_1) + \theta_0 \subseteq K$  such that the semiline

$$\{(x_2, y_2) + \lambda((\alpha, t) - (x_2, y_2)) : \lambda \geq 0\}$$

cuts the interval  $[(\alpha, v), (u, \beta)]$  in a point  $(s_0, t_0)$ . Since  $(\alpha, t)$  is a convex combination of  $(x_2, y_2)$  and  $(s_0, t_0)$  we have that  $L_1$  is included in  $K$ .

Repeating the same argument for  $L_2$ , we obtain that it is also included in  $K$ , and since  $K$  is convex we obtain that  $\text{co}(L_1 \cup L_2) \subseteq K$ .  $\square$

Throughout the rest of this section,  $K$  will always denote a  $q$ -compact convex subset in  $(\mathbb{R}^2, q)$ . In order to complete the geometric information about the structure of  $K$ , let us define the following sets.

$$\begin{aligned} \Delta_K &:= \text{co}\{(\alpha, \beta), (\alpha, v), (u, \beta)\} \\ S_K &:= K \cap (\mathbb{R} \times [\beta, +\infty)) \cap ([\alpha, +\infty) \times \mathbb{R}) \end{aligned}$$

and

$$R_K := K \cap H$$

where  $H$  is the closed half upper plane determined by the line containing the segment  $[(u, \beta), (\alpha, v)]$ . Observe that  $R_K$  and  $S_K$  are both convex sets contained in the rectangle  $[\alpha, u] \times [\beta, v]$  and therefore, they are  $q^s$ -bounded.

Finally, let us consider the set

$$F_K := \begin{cases} \partial_{q^s}(\overline{R_K}^{q^s}) \setminus ((u, \beta), (\alpha, v)), & \text{if } R_K \neq [(u, \beta), (\alpha, v)] \\ [(u, \beta), (\alpha, v)], & \text{if } R_K = [(u, \beta), (\alpha, v)] \end{cases}$$

where  $\overline{R_K}^{q^s}$  is the  $q^s$ -closure of  $R_K$  and  $\partial_{q^s}$  denotes the border respect to the topology determined by  $q^s$ . Since  $R_K$  is convex, the set  $F_K$  is a simple arc. Furthermore, we have the following contentions

$$R_K \subset S_K \subset K.$$

Additionally, since  $K$  is convex and  $F_K$  is an arc, it follows from the definition of  $(\alpha, v)$  and  $(u, \beta)$ , that  $F_K$  cannot contain any horizontal or vertical non-trivial segment.

If  $K$  has non empty interior, by Lemma 3 we also have that  $\Delta_K \subseteq S_K$ .

**Lemma 4.**  $F_K \subseteq K$ .

*Proof.* Let  $(x_1, y_1) \in F_K \subseteq \overline{R_K}^{q^s}$  and assume that it does not belong to  $K$ . Let  $U_t := (-\infty, x_1 - t) \times \mathbb{R}$  and  $V_t := \mathbb{R} \times (-\infty, y_1 - t)$ ,  $t > 0$ . Both of them are  $q$ -open. Since  $F_K$  does not contain either an horizontal or a vertical segment, we have that the sets  $\mathcal{U} := \{U_t, V_t\}_{t>0}$  constitute a  $q$ -open cover for  $K$ , and so there are  $t_1, t_2$  such that  $K \subseteq U_{t_1} \cup V_{t_2}$ .

Moreover,  $(x_1, y_1) \in (x_1 - t_1, +\infty) \times (y_1 - t_2, +\infty)$ , which is a  $q^s$ -open set that does not intersect  $K$ . This is a contradiction with the fact that  $(x_1, y_1)$  is in  $\overline{R_K}^{q^s} \subset \overline{K}^{q^s}$ .  $\square$

**Lemma 5.** *The set  $R_K$  is a  $q^s$ -compact convex set. Additionally, if  $K$  has non empty interior, then  $S_K$  is  $q^s$ -compact too.*

*Proof.* If  $S_K$  is a singleton then  $R_K = S_K$ , and the result is obvious. Assume that  $S_K$  is not a singleton. Clearly  $R_K$  is convex and  $q^s$ -bounded. So, we only need to prove that it is  $q^s$ -closed. This is equivalent to proving that  $\partial_{q^s}(R_K) \subset R_K$ .

By the definition of  $R_K$ , we have that

$$\partial_{q^s}(R_K) = F_K \cup T,$$

where  $T$  is the segment  $T = [(\alpha, v), (u, \beta)]$ . Since  $K$  is convex and the points  $(\alpha, v), (u, \beta)$  belong to  $K$ , it follows that  $T \subset R_K \subset K$ .

Consider the closed half plane  $H$  determined by the segment  $T$  which does not contain the point  $(\alpha, \beta)$ . Since  $H$  is closed, and  $R_K = H \cap K \subset H$ , it follows that  $F_K \subset H$ . This fact, in combination with Lemma 4 yields that  $F_K \subset R_K$ . Hence we can conclude that  $\partial_{q^s}(R_K) = F_K \cup T \subset R_K$  and thus  $R_K$  is  $q^s$ -closed.

Now assume that  $K$  has non-empty interior. In this case, we can use Lemma 3 to deduce that

$$\partial_{q^s}(S_K) = F_K \cup \{[(\alpha, \beta), (u, \beta)) \cup ((\alpha, v), (\alpha, \beta))]\}.$$

By the same lemma, we have that

$$[(\alpha, \beta), (u, \beta)) \cup ((\alpha, v), (\alpha, \beta))] \subset S_K.$$

Now, by Lemma 4, it is geometrically obvious that  $F_K \subset S_K$  and therefore  $\partial_{q^s}(S_K) \subset S_K$ . Since  $S_K$  is a  $q^s$ -bounded set, we have the wanted result.  $\square$

**Theorem 6.** *Every  $q$ -compact convex set  $K$  in  $(\mathbb{R}^2, q)$  is strongly  $q$ -compact.*

*Proof.* By Lemma 5, the set  $R_K$  is  $q^s$ -compact and satisfies  $R_K \subset K$ . Let us prove that  $K \subset R_K + \theta_0$ . First observe that

$$K \setminus R_K \subset \Delta_K \cup \{(\alpha, v) + \theta_0, (u, \beta) + \theta_0\}.$$



On the other hand, since the points  $(\alpha, v)$  and  $(u, \beta)$  belong to  $R_K$ , it follows that  $(\alpha, v) + \theta_0 \subset R_K + \theta_0$  and  $(u, \beta) + \theta_0 \subset R_K + \theta_0$ . It only rests to prove that  $\Delta_K \subset R_K + \theta_0$ , but this is obvious since  $R_K + \theta_0$  is convex and the point  $(\alpha, \beta) \in (\alpha, v) + \theta_0 \subset R_K + \theta_0$ . Thus

$$\Delta_K = \text{co}\{(\alpha, \beta), (\alpha, v), (u, \beta)\} \subset R_K + \theta_0.$$

Now the proof is complete.  $\square$

**Remark 7.** In [7, Theorem 4.7] it was proved that for the  $q^s$ -closed subsets of  $\mathbb{R}^2$ , the notions of  $q$ -compactness and strong  $q$ -compactness coincide. However, as mentioned in Section 2, even in  $\mathbb{R}^2$  there are examples of (non  $q^s$ -closed)  $q$ -compact sets that are not strongly  $q$ -compact. Since each strongly  $q$ -compact set is  $q$ -compact (see [2, Proposition 11]), our result implies that for the case of  $q$ -convex sets this result is also true, even if it is not  $q^s$ -closed.

#### 4. THE STRUCTURE OF $q$ -CONVEX BODIES

For the rest of the paper, the letter  $K$  will always denote a  $q$ -convex body. Let us also recall that, if  $x, y \in \mathbb{R}^2$ ,  $\{x, y\}$  means either the closed interval  $[x, y]$  or the left open interval  $(x, y]$ .

**Remark 8.** Let  $K \subset \mathbb{R}^2$  be a  $q$ -convex body. Consider the corresponding points  $(\alpha, v)$  and  $(u, \beta)$  computed just as in Section 3. Note that  $\beta = v$  if and only if  $\alpha = u$ . In this case, we have that  $F_K = \{(u, v)\} \subseteq K$ , which implies necessarily that

$$K \subseteq (u, v) + \theta_0 \subseteq \overline{K}^{q^s}.$$

For instance, a set such as  $(x_0, y_0) + (\theta_0)^o$  defined by a single point  $(x_0, y_0)$  plus the  $q$ -interior  $(\theta_0)^o$  of  $\theta_0$  is a  $q$ -compact set; this is easy to verify. This implies that for the case  $\beta = v$ , the  $q$ -convex body  $K$  is necessarily

$$K = \{(u, t_0), (u, v)\} \cup \{(s_0, v), (u, v)\} \cup ((u, v) + (\theta_0)^o),$$

for some  $-\infty \leq t_0 \leq v$  and  $-\infty \leq s_0 \leq u$ , where  $\{x, y\}$  means either a left closed interval or a left open interval with limits  $x, y \in \mathbb{R}^2$ . Such a set may not be homeomorphic to the asymmetric  $q^s$ -closed unit ball  $\overline{B}_q^{q^s}$  in  $\mathbb{R}^2$ , since in this case the natural identification of the boundaries from  $K$  to  $\overline{B}_q^{q^s}$  may not be surjective (for example if  $t_0 = v$  and  $s_0 = u$ ).

Everything is now prepared to prove the main result of this section.

**Theorem 9.** Let  $K$  be a subset of the asymmetric Euclidean space  $(\mathbb{R}^2, q)$ . The following statements are equivalent.



(i)  $K$  is a  $q$ -convex body.

(ii) (1) Either there is a point  $(u, v) \in \mathbb{R}^2$  such that

$$K = \{(u, v), (u, t_0)\} \cup \{(s_0, v), (u, v)\} \cup ((u, v) + (\theta_0)^o)$$

for some  $-\infty \leq t_0 \leq v$  and  $-\infty \leq s_0 \leq u$ ,

(2) or there are real scalars  $s_0 \leq \alpha < u$ ,  $t_0 \leq \beta < v$  and a  $q^s$ -compact convex set  $K_0$  satisfying that

$$\text{co}\{(\alpha, v), (\alpha, \beta), (u, \beta)\} \subseteq K_0 \subseteq \text{co}\{(\alpha, v), (\alpha, \beta), (u, \beta), (u, v)\},$$

$$K_0 \subset K \subset K_0 + \theta_0$$

and such that

$$K = K_0 \cup ((\alpha, v) + (\theta_0)^o) \cup ((u, \beta) + (\theta_0)^o) \cup \{(u, t_0), (u, \beta)\} \cup \{(s_0, v), (\alpha, v)\}.$$

*Proof.* (i)  $\Rightarrow$  (ii) Take a  $q$ -convex body  $K$  and compute the numbers  $(\alpha, v)$  and  $(u, \beta)$  as explained in Section 3. If the points  $(\alpha, v)$  and  $(u, \beta)$  corresponding to  $K$  are the same, we are in the case explained in Remark 8, that leads to case (1). If  $(\alpha, v) \neq (u, \beta)$ , let  $K_0 := S_K$ , where  $S_K$  is the set described in Section 3. Obviously the set  $S_K$  satisfies

$$\text{co}\{(\alpha, v), (\alpha, \beta), (u, \beta)\} \subseteq S_K \subseteq \text{co}\{(\alpha, v), (\alpha, \beta), (u, \beta), (u, v)\}.$$

By Lemma 5,  $S_K$  is  $q^s$ -compact and by Theorem 6, we have that

$$S_K \subset K \subset R_K + \theta_0 \subset S_K + \theta_0.$$

Moreover, the proof of Lemma 3. gives that

$$S_K \cup ((\alpha, v) + (\theta_0)^o) \cup ((u, \beta) + (\theta_0)^o) \subseteq K \subset S_K + \theta_0.$$

Consider the rays  $T_1 = ((-\infty, v), (\alpha, v)]$  and  $T_2 = ((u, -\infty), (u, \beta)]$  and note that

$$S_K \cup ((\alpha, v) + (\theta_0)^o) \cup ((u, \beta) + (\theta_0)^o) \cup T_1 \cup T_2 = S_K + \theta_0.$$

This implies that every point  $x \in K$  that is not in  $S_K \cup ((\alpha, v) + (\theta_0)^o) \cup ((u, \beta) + (\theta_0)^o)$  must be in one of rays  $T_1$  or  $T_2$ . Observe that  $K$ ,  $T_1$  and  $T_2$  are convex sets, and so  $K \cap T_1$  and  $K \cap T_2$  are convex subsets of  $T_1$  and  $T_2$  containing the points  $(\alpha, v)$  and  $(u, \beta)$ , respectively. Since the only convex subsets of  $T_1$  are the segments, it follows that  $T_1 \cap K = \{(s_0, v), (\alpha, v)\}$  for some  $-\infty \leq s_0 \leq \alpha$ . Analogously, there exists  $-\infty \leq t_0 \leq \beta$  such that  $T_2 \cap K = \{(u, t_0), (u, \beta)\}$ . This proves the first implication.

(ii)  $\Rightarrow$  (i) It follows from the structure of  $K$  that it has non empty interior (for instance the point  $(\alpha, \beta) \in K$  is an interior point). If the set  $K$  is as in (1), it is obviously  $q$ -compact since each open set containing the point  $(u, v)$  contains the whole set  $K$ . If the set  $K$  is as

in (2), then it is strongly  $q$ -compact and thus also  $q$ -compact. Now the proof is complete.  $\square$

By Theorem 1 we know that the  $q^s$ -closure of a  $q$ -convex body  $K \subset \mathbb{R}^2$  is  $q^s$ -homeomorphic to  $\overline{B_1^q(0)}^{q^s}$ . Using the geometric description of the  $q$ -convex bodies of the asymmetric space  $(\mathbb{R}^2, q)$  given in Theorem 9 we get the following

**Corollary 10.** *Let  $K$  be a  $q$ -convex body. It is then  $q^s$ -homeomorphic to a convex set  $K' \subseteq \overline{B_1^q(0)}^{q^s}$  that can be written as*

$$K' = B_1^q(0) \cup C$$

where  $C$  is a connected subset of the  $q^s$ -boundary of  $\overline{B_1^q(0)}^{q^s}$ .

**Final Remark.** It is well known that convex sets in  $\mathbb{R}^2$  satisfy properties that are not generally true for convex sets in higher dimensions. The same happens in the asymmetric case. For instance, Theorem 6, is no longer true in  $n$ -dimensional asymmetric normed lattices if  $n > 2$  (an example of this situation is explained in [13]). This is why we decided to work the 2-dimensional case separately in this note. To see more about the geometric structure of compact convex sets in asymmetric normed spaces, the reader can consult [13].

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[Natalia Jonard-Pérez] Departamento de Matemáticas, Facultad de Matemáticas, Campus Espinardo, 30100 Murcia, Spain, e-mail: nataliajonard@gmail.com

[Enrique A. Sánchez Pérez] Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain, e-mail: easancpe@mat.upv.es